Dirichlet Series I

Talip Can TERMEN

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İzmir Institute of Technology

Definition

Let $f : \mathbb{N} \to \mathbb{C}$ be an arithmetical function. Then the series of the form

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

are called Dirichlet series.

- We let s be a complex variable and write

$$s = \sigma + it.$$

Then, we have $|n^s| = n^{\sigma}$.

Dirichlet Series

The set of points $s = \sigma + it$ such that $\sigma \geq a$ is called a half-plane. We will show that for each Dirichlet series there is a half-plane $\sigma > \sigma_c$ in which the series converges, and another half-plane $\sigma > \sigma_a$ in which it converges absolutely.

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Observation. If $\sigma > a$, we have

$$|n^s| = n^\sigma \geq n^a$$

. Hence

$$\left|\frac{f(n)}{n^s}\right| \leq \frac{|f(n)|}{n^a}.$$

Threfore if a Dirichlet series converges absolutely for $s = a + ib$, then it converges absolutely for all s with $\sigma \geq a$.

Dirichlet Series

Theorem1

Suppose the series $\sum_{n=1}^{\infty} |f(n)n^{-s}|$ does not converge for all s or diverge for all s. Then there exists a real number σ_a, called the abscissa of absolute convergence, such that the series $\sum_{n=1}^{\infty} f(n)n^{-s}$ converges absolutely if $\sigma > \sigma_a$ but does not converge absolutely if $\sigma < \sigma_a$.

Theorem1

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Note. If $\sum_{n=1}^{\infty} |f(n)n^{-s}|$ converges everywhere, then $\sigma_a = -\infty$. $\sum_{n=1}^{\infty} |f(n)n^{-s}|$ diverges everywhere, then $\sigma_a = \infty$.

Examples

- Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

  converges absolutely for $\sigma > 1$ When $s = 1$ the series diverges, so $\sigma_a = 1$
- The series $\sum_{n=1}^{\infty} \frac{n^n}{n^s}$ diverges for every s so $\sigma_a = \infty$.
- The series $\sum_{n=1}^{\infty} \frac{n^{-n}}{n^s}$ converges for every s so $\sigma_a = -\infty$.

- If f is bounded, say $|f(n)| \leq M$ for all $n \geq 1$, then $\sum f(n)n^{-s}$ converges absolutely for $\sigma > 1$, so $\sigma_a \leq 1$.

$$|\sum f(n)n^{-s}| \leq M \sum \frac{1}{n^s}$$

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In particular , if χ is a Dirichlet character, the L-series

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

converges absolutely for $\sigma > 1$.

Lemma1

If $N \geq 1$ and $\sigma \geq c \geq \sigma_a$, we have

$$\left| \sum_{n=N}^{\infty} f(n) n^{-s} \right| \leq N^{-(\sigma - c)} \sum_{n=N}^{\infty} |f(n)| n^{-c}.$$

Assume that $\sum f(n)n^{-s}$ converges absolutely for $\sigma > \sigma_a$, and let $F(s)$ denote the sum function

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \qquad \text{for } \sigma > \sigma_a.$$

Theorem2

We have

$$\lim_{\sigma \to \infty} F(\sigma + it) = f(1)$$

uniformly for $-\infty < t < \infty$.

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- $\zeta(\sigma + it) \to 1$ as $\sigma \to \infty$
- $L(\sigma + it, \chi) \to 1$ as $\sigma \to \infty$

Uniqueness Theorem

Theorem3

Given two Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \quad \text{and} \quad G(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$$

both absolutely convergent for $\sigma > \sigma_a$. If $F(s) = G(s)$ for each s in an infinite sequence $\{s_k\}$ such that

$$\sigma_k \to \infty \quad \text{as} \quad k \to \infty$$

then

$$f(n) = g(n)$$

for every n.

Theorem4

Let $F(s) = \sum f(n)n^{-s}$ and asuume $F(s) \neq 0$ for some s with $\sigma > \sigma_a$. Then there is a half-plane $\sigma > c > \sigma_a$ in which $F(s)$ is never zero.

Theorem5

Given two functions $F(s)$ and $G(s)$ represented by Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \quad \text{and} \quad G(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$$

for $\sigma > a$ for $\sigma > b$.

Then in the half-plane where both series converges absolutely, we have

$$F(s)G(s) = \sum_{n=1}^{\infty} \frac{h(n)}{n^s},$$

where $h = f * g$, the Dirichlet convolution of f and g. Conversely, if $F(s)G(s) = \sum \alpha(n)n^{-s}$ for all s in a sequence $\{s_k\}$ with $\sigma_k \to \infty$ as $k \to \infty$, then

$$\alpha = f * g.$$

Examples

- Both series $\sum n^{-s}$ and $\sum \mu(n)n^{-s}$ converge absolutely for $\sigma > 1$. Taking $f(n) = 1$ and $g(n) = \mu(n)$, we find

$$h(n) = 1 * \mu(n) = \left\lfloor \frac{1}{n} \right\rfloor = \begin{cases} 1 & , n = 1 \\ 0 & , n \neq 1 \end{cases}$$

so

$$\zeta(s) \sum_{n=1}^{\infty} \mu(n) n^{-s} = \sum_{n=1}^{\infty} \frac{1 * \mu(n)}{n^s} = 1$$

In particular, this shows that $\zeta(s) \neq 0$ for $\sigma > 1$.

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)} \quad \text{if } \sigma > 1.$$

Examples

- More generally, assume $f(1) \neq 0$ and let $g = f^{-1}$, the Dirichlet inverse of f. Then in any half-plane where both series $F(s)$ and $G(s)$ converge absolutely, we have

$$F(s) \neq 0 \text{ and } G(s) = \frac{1}{F(s)}.$$

$$F(s)G(s) = \sum_{n=1}^{\infty} \frac{(f * g)(n)}{n^s} = \sum_{n=1}^{\infty} \frac{I(n)}{n^s} = 1,$$

where

$$f * f^{-1}(n) = I(n) = \left\lfloor \frac{1}{n} \right\rfloor.$$

Examples

- Assume $F(s) = \sum_{n=1}^{\infty} f(n)n^{-s}$ converges absolutely for $\sigma > \sigma_a$. If f is completely multiplicative, we have $f^{-1}(n) = \mu(n)f(n)$. Since

$$|f^{-1}(n)| = |\mu(n)||f(n)| \leq |f(n)|$$

the series $\sum_{n=1}^{\infty} f^{-1}(n)n^{-s}$ also converges absolutely for $\sigma > \sigma_a$ and we have

$$\sum_{n=1}^{\infty} \frac{\mu(n)f(n)}{n^s} = \sum_{n=1}^{\infty} \frac{f^{-1}(n)}{n^s} = \frac{1}{F(s)} \text{ if } \sigma > \sigma_a.$$

In particular, for every Dirichlet character χ we have

$$\sum_{n=1}^{\infty} \frac{\mu(n)\chi(n)}{n^s} = \frac{1}{L(s,\chi)}.$$

Examples

- Take $f(n) = u(n) = 1$, and $g(n) = \varphi(n)$, Euler's totient. Since $\varphi(n) \le n$, the series $\sum \varphi(n) n^{-s}$ converges absolutely for $\sigma > 2$. Also,
$$h(n) = \sum_{d|n} \varphi(d) = n$$
Then
$$\zeta(s) \sum \frac{\varphi(n)}{n^s} = \sum \frac{1 * \varphi(n)}{n^s} = \sum \frac{n}{n^s} = \zeta(s-1) \text{ if } \sigma > 2.$$
Therefore
$$\sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)} \text{ if } \sigma > 2.$$

Examples

- Take $(f) = 1$ and $g(n) = n^{\alpha}$. Then

$$\zeta(s)\zeta(s-\alpha) = \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{n=1}^{\infty} \frac{n^{\alpha}}{n^s} = \sum_{n=1}^{\infty} \frac{\sigma_{\alpha}(n)}{n^s}$$

if $\sigma > \max\{1, 1 + \mathrm{Re}(\alpha)\}$.

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  if $\sigma > \max\{1, 1 + \mathrm{Re}(\alpha)\}$.

- Take $(f) = 1$ and $g(n) = \lambda(n)$, Lioville's function. Then

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \frac{\zeta(2s)}{\zeta(s)} \text{ if } \sigma > 1.$$

Euler Product

Theorem 6

Let f be a multiplicative arithmetical function such that the series $\sum f(n)$ is absolutely convergent . Then the sum of the series can be expressed as an absolutely convergent infinite procut,

$$\sum_{n=1}^{\infty} f(n) = \prod_{p} \{1 + f(p) + f(p^2) + ...\}$$

extended over all primes.
Note that if f is completely multiplicative, the product simplifies and we have

$$\sum_{n=1}^{\infty} = \prod_{p} = \frac{1}{1 - f(p)}.$$

Euler Product

Theorem7

Assume $\sum f(n)n^{-s}$ converges absolutely for $\sigma > \sigma_a$. If f is multiplicative, we have

$$\sum_{n=1}^{\infty} f(n)n^{-s} = \prod_{p}\{1 + \frac{(p)}{p^s} + \frac{f(p^2)}{p^{2s}} + ...\} \text{ if } \sigma > \sigma_a.$$

and if f is completely multiplicative, we have

$$\sum_{n=1}^{\infty} f(n)n^{-s} = \prod_{p} \frac{p^s}{p^s - f(p)}.$$

Examples

Taking $f(n) = 1, \mu(n), \varphi(n), \sigma_\alpha(n), \lambda(n), \chi(n)$, respectively, we obtain the following Euler products:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1 - p^{-s}} \text{ if } \sigma > 1.$$

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \prod_p (1 - p^{-s}) \text{ if } \sigma > 1.$$

$$\frac{\zeta(s-1)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s} = \prod_p \frac{1 - p^{-s}}{1 - p^{1-s}} \text{ if } \sigma > 2.$$

Examples

$$\zeta(s)\zeta(s-\alpha) = \sum_{n=1}^{\infty} \frac{\sigma_\alpha(n)}{n^s} = \prod_p \frac{1}{(1-p^{1-s})(1-p^{\alpha-s})}$$

if $\sigma > \max\{1, 1+\mathrm{Re}(\alpha)\}$.

$$\frac{\zeta(2s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \prod_p \frac{1}{1+p^{-s}} \quad \text{if } \sigma > 1.$$

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \frac{1}{1-\chi(p)p^{-s}} \quad \text{if } \sigma > 1.$$

Euler Product

If $\chi = \chi_1$, the principal character mod k, then $\chi_1(p) = 0$, if $p|k$, and $\chi_1(p) = 1$ if $p \nmid k$. So the Euler product for $L(s,\chi)$ becomes

$$L(s,\chi_1) = \prod_{p \nmid k} \frac{1}{1-p^{-s}} = \prod_{p} \frac{1}{1-p^{-s}} \prod_{p|k}(1-p^{-s}) = \zeta(s) \prod_{p|k}(1-p^{-s}).$$

Lemma2

Let $s_0 = \sigma_0 + it_0$ and assume that the Dirichlet series $\sum f(n)n^{-s}$ has bounded partial sums, say

$$\left|\sum_{n\leq x} f(n)n^{-s}\right| \leq M$$

for all $x \geq 1$. Then for each s with $\sigma > \sigma_0$, we have

$$\left|\sum_{a<n\leq b} f(n)n^{-s}\right| \leq 2Ma^{\sigma_0-\sigma}\left(1 - \frac{|s-s_0|}{\sigma-\sigma_0}\right)$$

Examples

- If the partial sums $\sum_{n \leq x} f(n)$ are bounded, Lemma2 implies that $\sum f(n)n^{-s}$ converges for $\sigma > 0$. In fact, if we take $s_0 = \sigma_0 = 0$, we obtain for $\sigma > 0$

$$\left| \sum_{a < n \leq b} f(n)n^{-s} \right| \leq Ka^{-\sigma}$$

where K is independent of a. Letting $a \to \infty$ we find that $\sum f(n)n^{-s}$ converges if $\sigma > 0$. In particular, this shows that the Dirichlet series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^s}$$

converges for $\sigma > 0$ since

$$\left| \sum_{n \leq x} (-1)^n \right| \leq 1.$$

Examples

- Similarly, if χ is any non-principal Dirichlet character mod$k$, we have

$$\left|\sum_{n\leq x}\chi(n)\right| \leq \varphi(k)$$

so

$$\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}$$

converges for $\sigma > 0$.

Dirichlet Series

Theorem8

If the series $\sum f(n)n^{-s}$ converges for $s = \sigma_0 + it_0$, then it also converges for all s with $\sigma > \sigma_0$. If it diverges for $s = \sigma_0 + it_0$, then it diverges for all s with $\sigma < \sigma_0$.

Theorem9

If the series $\sum f(n)n^{-s}$ does not converge everywhere or diverge everywhere, then there exists a real number σ_c, called the abscissa of convergence, such that the series converges for all s in the half-plane $\sigma > \sigma_c$ and diverges for all s in the half-plane $\sigma < \sigma_c$.

Note. If the series converges everywhere we define $\sigma_c = -\infty$, and if it converges nowhere we define $\sigma_c = \infty$.
Since absolute convergence implies convergence, we always have $\sigma_a \geq \sigma_c$. If $\sigma_a > \sigma_c$ there is an infinite strip

$$\sigma_c < \sigma < \sigma_a$$

in which the series converges conditionally.

Theorem10

For any Dirichlet series with σ_c finite, we have

$$0 \leq \sigma_a - \sigma_c \leq 1.$$

Example

The series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^s}$$

converges if $\sigma > 0$, but the convergence is absolute only if $\sigma > 1$. Therefore, $\sigma_c = 0$ and $\sigma_a = 1$.

References

1. Apostol, T.M., Introduction to analytic number theory. Springer, 1976.